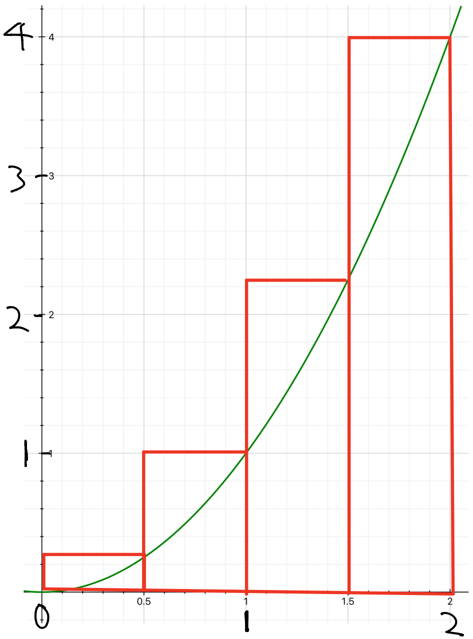
Math 1510 Week 12

Average Value of a function

eg. What is the average value of

$f(x) = x^2$ on $[0,2]$?



Approximation 1 (4 sample points)

$$\text{Average} \approx \frac{1}{4}\left[f(0.5)+f(1)+f(1.5)+f(2)\right]$$
$$= \frac{1}{2}\left[f(0.5)\cdot 0.5 + f(1)\cdot 0.5 + f(1.5)\cdot 0.5 + f(2)\cdot 0.5\right]$$

Approximation 2 (8 sample points)

$$\text{Average} \approx \frac{1}{8}\left[f(0.25)+f(0.5)+\dots+f(2)\right]$$
$$= \frac{1}{2}\left[f(0.25)\cdot 0.25 + f(0.5)\cdot 0.25) + \dots + f(2)\cdot 0.25\right]$$

Similarly, for n sample points:

height of rectangle

length

$$\text{Average} \approx \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{2k}{n}\right) = \frac{1}{2}\sum_{k=1}^{n} f\left(\frac{2k}{n}\right)\cdot\frac{2}{n}$$

Take $n \to \infty$:

2 = length of $[0,2]$

$$\text{Average} = \frac{1}{2}\int_0^2 f(x)\,dx = \frac{1}{2}\int_0^2 x^2\,dx = \left[\frac{1}{6}x^3\right]_0^2 = \frac{4}{3}$$

For a general $f(x)$ defined on $[a,b]$

$$\boxed{\text{Average value of } f(x) = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

Mean Value Theorem for Integration

Suppose $f(x)$ is continuous on $[a,b]$,

then $\exists\, c \in (a,b)$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

function value at c

Average value of $f(x)$ on $[a,b]$

Meaning:

The average value of $f(x)$ on $[a,b]$ is achieved at some $c \in (a,b)$

eg $f(x) = x^2$ on $[0,2]$

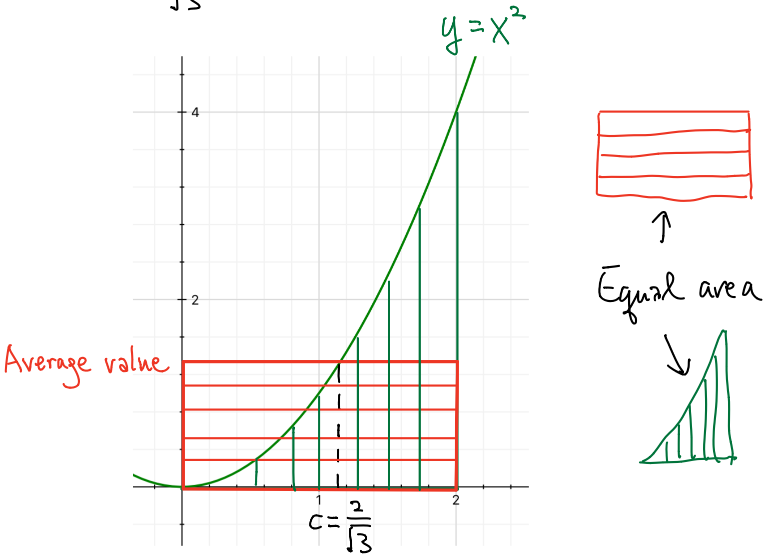
Find the "c" in the MVT for integration

Sol $a=0$, $b=2$. For $c \in (0,2)$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx,$$

$$c^2 = \frac{1}{2-0}\int_0^2 x^2\,dx = \frac{1}{2}\left[\frac{1}{3}x^3\right]_0^2 = \frac{4}{3}$$

$$\Rightarrow c = \frac{2}{\sqrt{3}}$$



Integration of Even / Odd functions

$$f(x) \text{ is } \begin{cases} \text{even} & \text{if } f(-x) = f(x), \\ \text{odd} & \text{if } f(-x) = -f(x) \end{cases} \quad \forall x \in \text{Domain}$$

Let $a > 0$.

① If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

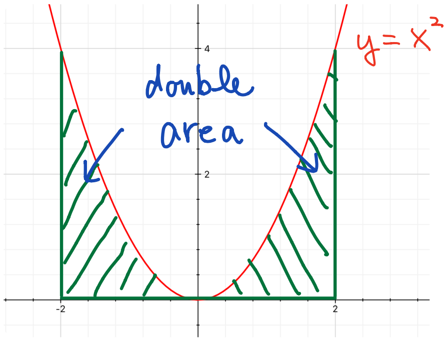
eg. Let $f(x) = x^2$

$f(-x) = (-x)^2 = x^2 = f(x) \Rightarrow f$ is even

$\therefore \int_{-2}^{2} x^2\,dx = 2\int_{0}^{2} x^2\,dx = \frac{2}{3}x^3\Big|_0^2 = \frac{16}{3}$

f is even

$\Rightarrow$ symmetric about y-axis



② If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$

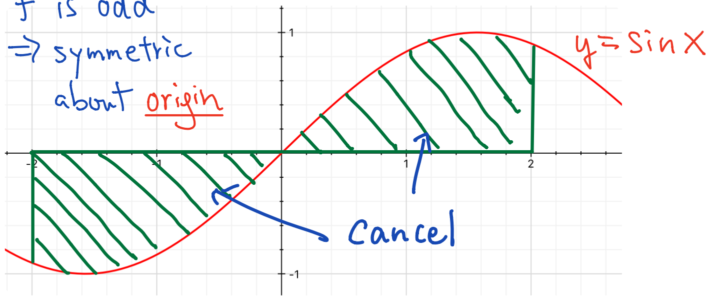
eg. Let $f(x) = \sin x$

$f(-x) = \sin(-x) = -\sin x = -f(x) \Rightarrow f$ is odd

$\therefore \int_{-2}^{2} \sin x = 0$

f is odd

$\Rightarrow$ symmetric about origin



$$\text{eg} \quad \int_{-1}^{1} \Big(\underbrace{|x|}_{\text{even}} + \underbrace{\tan x + \frac{x^3}{\cos x}}_{\text{odd}} \Big)\,dx$$

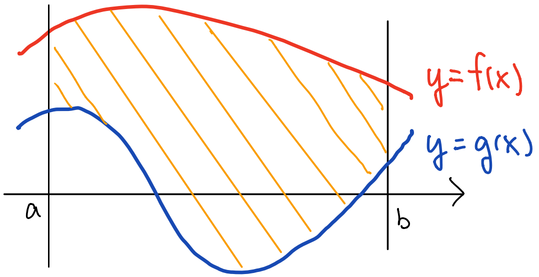
$= 2\int_0^1 |x|\,dx + 0 + 0$

$= 2\int_0^1 x\,dx = x^2\Big|_0^1 = 1^2 - 0^2 = 1$

Area

eg 1

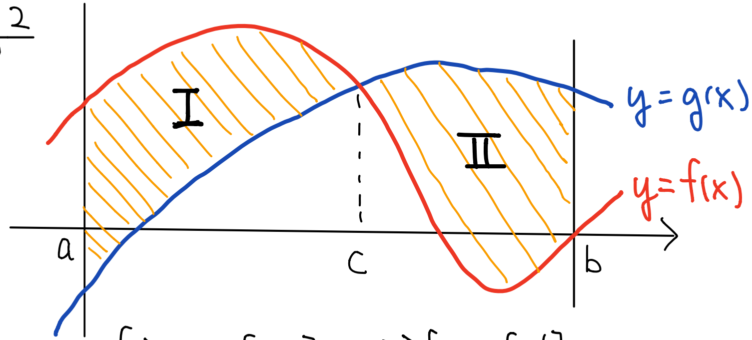
Suppose $f(x) \geq g(x)$ on $[a, b]$



Area of shaded region

$$= \int_a^b (f(x) - g(x))\,dx$$

eg 2



$f \geq g$ on $[a,c]$ $\quad g \geq f$ on $[c,b]$

Area of shaded region = Area of I + Area of II

$$= \int_a^c \underbrace{(f(x) - g(x))}\,dx + \int_c^b \underbrace{(g(x) - f(x))}\,dx$$

Both integrands $= |f(x) - g(x)|$ on their intervals of integration

In general, if $a \leq b$,

Area between $y = f(x)$ and $y = g(x)$ over $[a, b]$

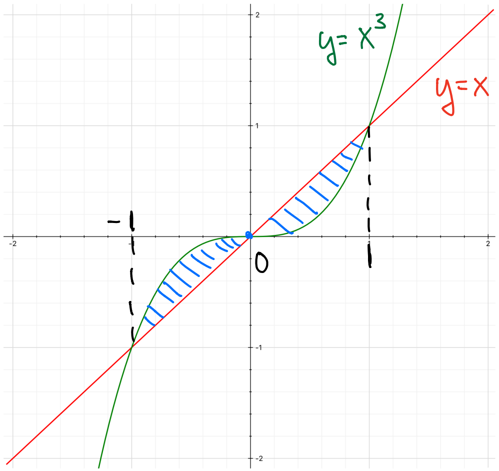
$$= \int_a^b |f(x) - g(x)|\,dx$$

eg Find area bounded between the curves

$y = x$ and $y = x^3$

Sol Find intersections :

$$x = x^3$$
$$x^3 - x = 0$$
$$x(x^2-1) = 0$$
$$x(x+1)(x-1) = 0$$
$$\therefore \; x = 0 \text{ or } \pm 1$$



$$\text{Area} = \int_{-1}^{0} (x^3 - x)\,dx + \int_0^1 (x - x^3)\,dx$$

$$= \left[\tfrac{1}{4}x^4 - \tfrac{1}{2}x^2\right]_{-1}^{0} + \left[\tfrac{1}{2}x^2 - \tfrac{1}{4}x^4\right]_0^1$$

$$= \left[0 - \left(-\tfrac{1}{4}\right)\right] + \left[\tfrac{1}{4} - 0\right]$$

$$= \frac{1}{2}$$

Alt. Sol By symmetry,

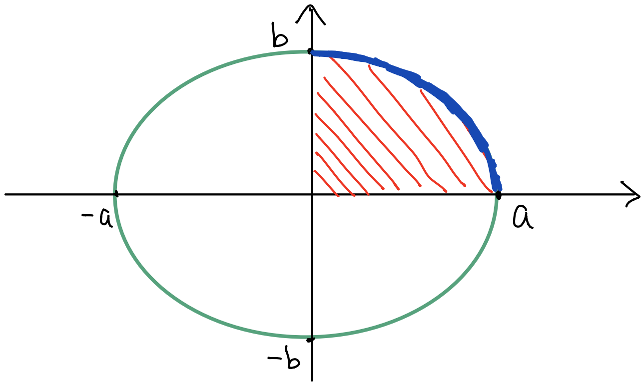
Area $= 2 \times$ Area of 

$$= 2\int_0^1 (x - x^3)\,dx$$

$$= 2\left(\tfrac{1}{4}\right)$$

$$= \frac{1}{2}$$

eg Let $a, b > 0$. Find area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

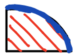


We consider the part in the first quadrant:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$y = + b\sqrt{1 - \frac{x^2}{a^2}}$$

because $y \geq 0$ in 1st quadrant

Area of 

Let $x = a\sin\theta$

$dx = a\cos\theta \, d\theta$

When $x = a$, $\theta = \frac{\pi}{2}$

$x = 0$ $\theta = 0$

$$= \int_0^a b\sqrt{1 - \frac{x^2}{a^2}} \, dx$$

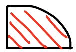
$$= \int_0^{\frac{\pi}{2}} b\sqrt{\frac{1 - a^2\sin^2\theta}{a^2}} \, a\cos\theta \, d\theta$$

$$= ab\int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta$$

$$= \frac{ab}{2}\int_0^{\frac{\pi}{2}} (1 + \cos 2\theta) \, d\theta$$

$$= \frac{ab}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{ab}{2}\left(\frac{\pi}{2} - 0\right) = \frac{\pi ab}{4}$$

Area of ellipse $= 4 \times$ Area of 

$= \pi ab$

Rmk $a = b = r \Rightarrow$ Area of circle $= \pi r^2$

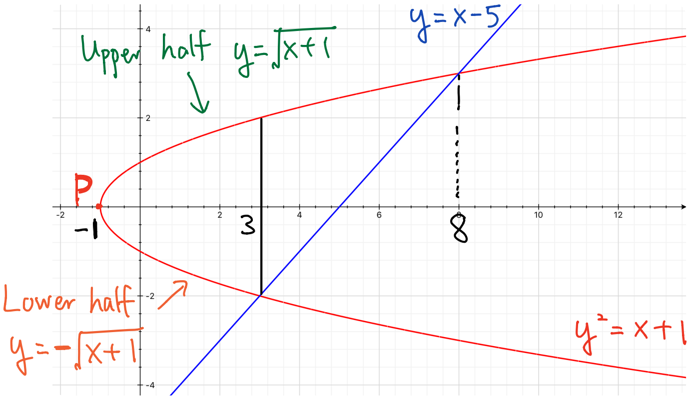
eg Find the area bouned between $y^2 = x+1$ and $y = x-5$

Sol **Method I** : Integrate $\underbrace{\text{w.r.t}}_{\text{with respect to}}$ x

At intersections: $(x-5)^2 = x+1$

$$x^2 - 11x + 24 = 0 \Rightarrow x = 3 \text{ or } 8$$

At P, $y=0 \Rightarrow 0^2 = x+1 \Rightarrow x = -1$



$$\text{Area} = \underbrace{\int_{-1}^{3}\left[\sqrt{x+1} - \left(-\sqrt{x+1}\right)\right]dx}_{\text{or } 2\int_{-1}^{3}\sqrt{x+1}\,dx \text{ by symmetry}} + \int_{3}^{8}\left[\sqrt{x+1} - (x-5)\right]dx$$

$\boxed{\text{Ex Compute!}}$

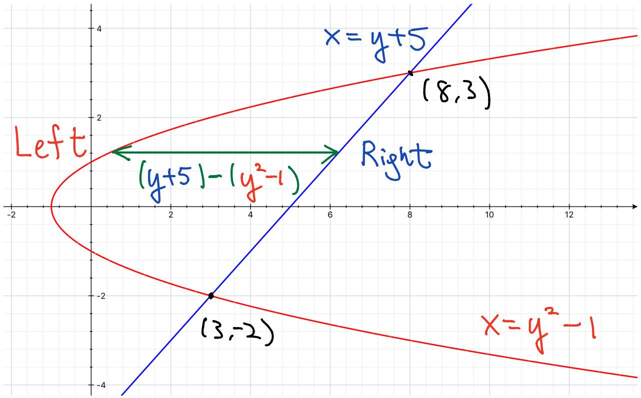
Method II: Integrate w.r.t. y

At intersections: $x = 3 \Rightarrow y = 3-5 = -2$

$x = 8 \Rightarrow y = 8-5 = 3$

Also, $y^2 = x+1 \iff x = y^2 - 1$

$y = x-5 \iff x = y+5$

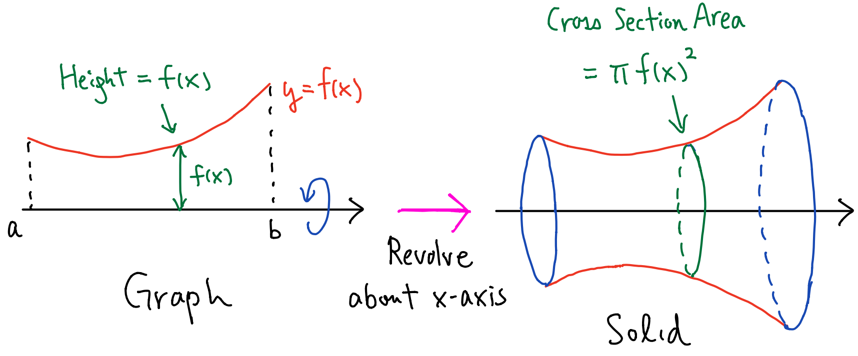


$$\text{Area} = \int_{-2}^{3} \overset{\text{Right}}{(y+5)} - \overset{\text{Left}}{(y^2-1)}\,dy$$

$$= \int_{-2}^{3} (-y^2 + y + 6)\,dy$$

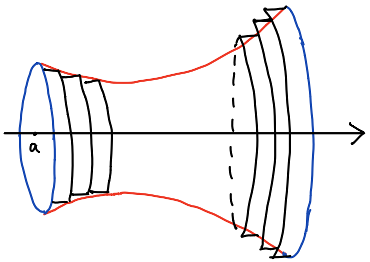
$$= \left[-\frac{1}{3}y^3 + \frac{1}{2}y + 6y\right]_{-2}^{3} = \frac{125}{6}$$

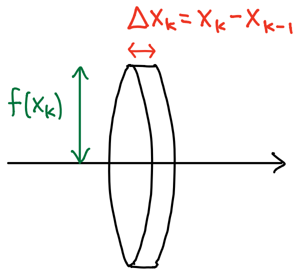
Solid of Revolution (Disk Method)



Q How to find the volume of the solid?

A Approximation using slices (disk)





Divide $[a,b]$ by points

$a = x_0 < x_1 < x_2 < \cdots < x_n = b$

Approximate solid by slices

Volume of k-th disk

$= \pi f(x_k)^2 \Delta x_k$

Volume of disks $= \pi \sum_{k=1}^{n} f(x_k)^2 \Delta x_k$

Take limit $n \to \infty$:

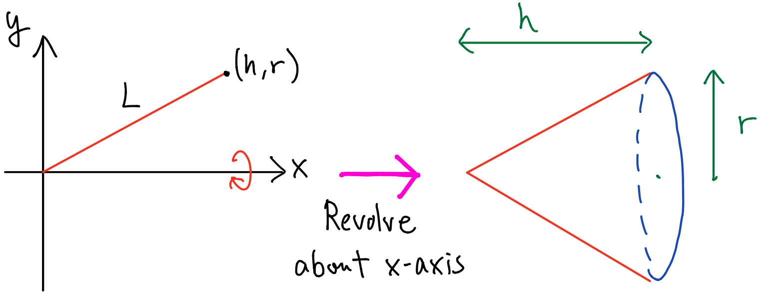
For the Solid of Revolution of $y=f(x)$ on $[a,b]$ about x-axis

$$\text{Volume} = \int_a^b \underbrace{\pi f(x)^2}_{\text{Cross Section Area}} dx$$

Integrate cross section area along length gives volume

Compare : Area $= \int_a^b f(x)\,dx$

Integrate height along length gives area eg Let $h, r > 0$



circular cone of base radius r and height h

Q Find volume of cone

Sol Equation of L: $\dfrac{y-0}{x-0} = \dfrac{r}{h} \Rightarrow y = \dfrac{r}{h}x$

$$\text{Volume} = \pi \int_0^h \left(\frac{r}{h}x\right)^2 dx$$

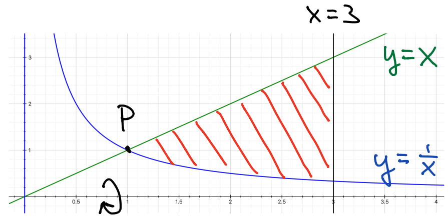
$$= \frac{\pi r^2}{h^2} \int_0^h x^2 dx$$

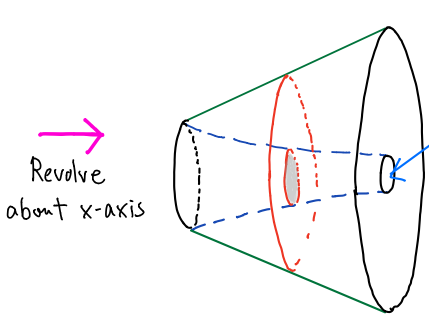
$$= \frac{\pi r^2}{h^2} \left[\frac{1}{3}x^3\right]_0^h$$

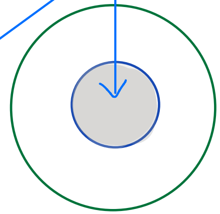
$$= \frac{\pi r^2 h}{3} \quad \left(\frac{1}{3} \times \text{base area} \times \text{height}\right)$$

$\boxed{\text{Ex Derive formula for volume of sphere}}$

eg







Cross section

For P,

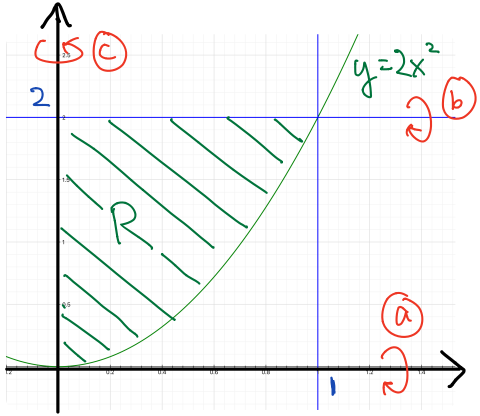
$x = \dfrac{1}{x} \Rightarrow x^2 = 1 \Rightarrow x = 1$ (-1 is rejected $\because x > 0$)

$$\therefore \text{Volume} = \pi \int_1^3 \left[x^2 - \underbrace{\left(\frac{1}{x}\right)^2}_{\text{subtract hollow part}}\right] dx$$

$$= \pi \left[\frac{x^3}{3} + \frac{1}{x}\right]_1^3$$

$$= \pi \left[\left(9 + \frac{1}{3}\right) - \left(\frac{1}{3} + 1\right)\right] = 8\pi$$

eg Revolving around different axes



Find volume of the solid obtained by revolving the region R about

a. x-axis b. the line $y=2$

c. y-axis

Sol

a. $V = \pi \int_0^1 \left[2^2 - (2x^2)^2\right] dx$

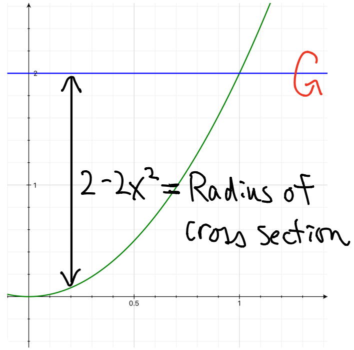
$= \pi \left[4x - \frac{4}{5}x^5\right]_0^1 = \frac{16\pi}{5}$

b. $V = \pi \int_0^1 (2-2x^2)^2 dx$

$= \pi \int_0^1 (4 - 8x^2 + 4x^4) dx$

$= \pi \left[4x - \frac{8}{3}x^3 + \frac{4}{5}x^5\right]_0^1$

$= \frac{32\pi}{15}$



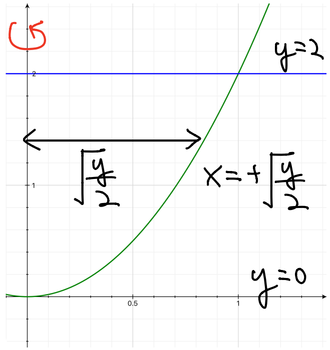
Rmk Rotating around a horizontal axis ⇒ Integrate w.r.t. x

c. $y = 2x^2 \Rightarrow x = \pm\sqrt{\frac{y}{2}}$

$V = \pi \int_0^2 \left(\sqrt{\frac{y}{2}}\right)^2 dy$

$= \frac{\pi}{2} \int_0^2 y\, dy$

$= \frac{\pi}{2} \left[\frac{1}{2}y^2\right]_0^2 = \pi$

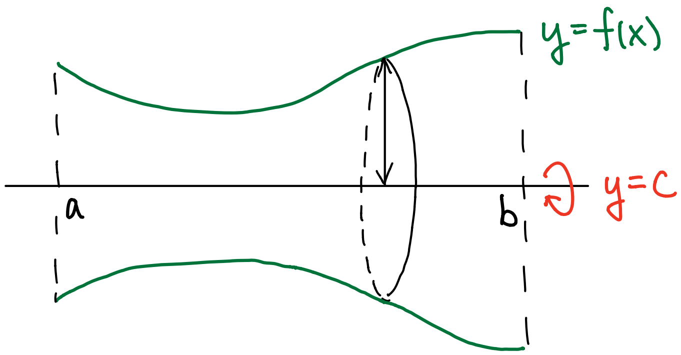


Rmk Rotating around a vertical axis ⇒ Integrate w.r.t. y

Summary for volume of Solid of Revolution

Horizontal axis $y=c$ ($c=0 \Rightarrow$ x-axis)

- Express curve as $y=f(x)$
- Integration variable : x



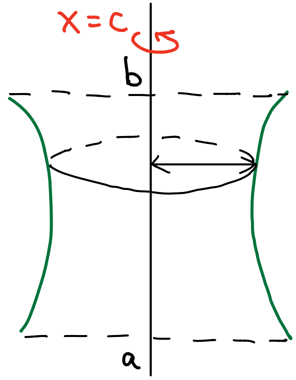
Cross Section: radius $= |f(x)-c|$

area $= (f(x)-c)^2$

$$\boxed{V = \pi\int_a^b (f(x)-c)^2\,dx \underset{\uparrow \text{ if } c=0 \text{ (x-axis)}}{=} \pi\int_a^b f(x)^2\,dx}$$

Vertical axis $x=c$ ($c=0 \Rightarrow$ y-axis)

- Express curve as $x=f(y)$
- Integration variable : y



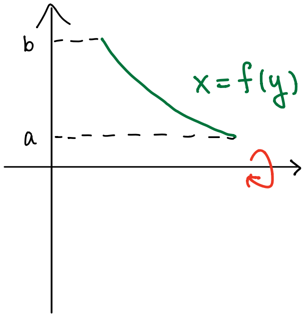
Cross Section: radius $= |f(y)-c|$

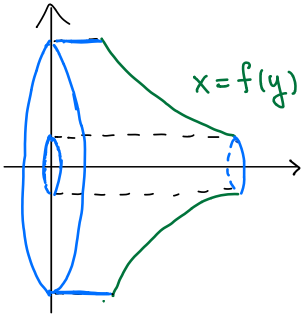
area $= (f(y)-c)^2$

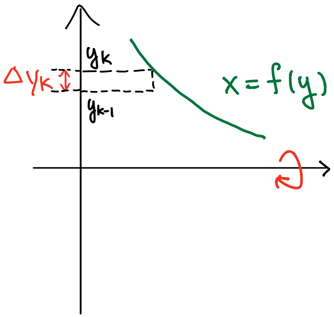
$$\boxed{V = \pi\int_a^b (f(y)-c)^2\,dy \underset{\uparrow \text{ if } c=0 \text{ (y-axis)}}{=} \pi\int_a^b f(y)^2\,dy}$$

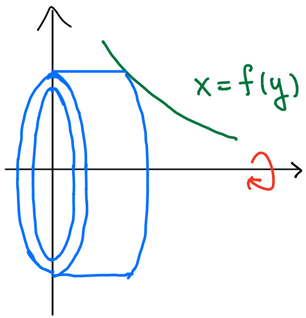
Shell Method (NOT FOR EXAM)

Also used to find volume of solid of revolution.









k-th horizontal slice

Volume of k-th shell

$$= \underbrace{\pi\left(y_k^2 - y_{k-1}^2\right)}_{\text{Cross Section Area}} \underbrace{f(y_k)}_{\text{width}}$$

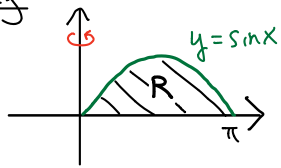
$$= \pi\,(y_k + y_{k-1})\, f(y)\, \Delta y$$

Volume of shells $= \sum_{k=1}^{n} \pi (y_k + y_{k-1}) f(y) \Delta y$

Take $n \to \infty \Rightarrow$ $\boxed{V = \int_a^b 2\pi y f(y)\, dy}$

$$\int_a^b \underbrace{2\pi y}_{\text{Circumference}} \underbrace{f(y)}_{\text{width}} \underbrace{dy}_{\text{thickness}} = \int_a^b \underbrace{f(y)}_{\text{width}}\, d(\underbrace{\pi y^2}_{\text{Cross Section area}})$$

eg



Shell method

$$V = \int_0^{\pi} 2\pi x \sin x\, dx$$

Disk method

$$V = \int_0^1 \left[\pi(\pi - \arcsin y)^2 - \pi(\arcsin y)^2\right] dy$$

$$= \int_0^1 \left(\pi^3 - 2\pi^2 \arcsin y\right) dy$$

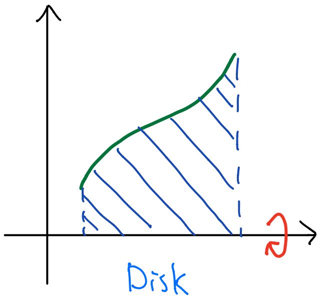
Rmk The curve is not a graph of a function of y

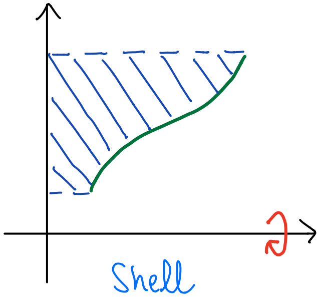
$\therefore$ Need to divide the curve into parts in disk method

Disk or Shell? (NOT FOR EXAM)

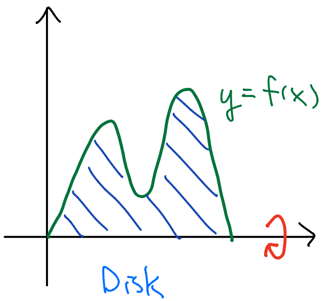
Points to Consider:

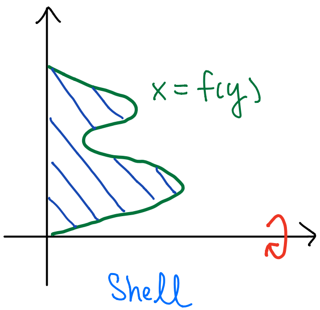
① How the region is cut





② function of x? y?

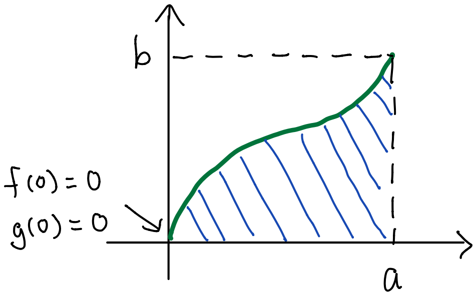




③ The function: f? f^{-1}? Easier to integrate?

Suppose the curve can be written as both

$y = f(x)$ and $x = g(y)$ $\quad (\therefore f^{-1} = g)$



One can show computationally the two methods give equal answers:

$$V = \int_0^b 2\pi y\,(a - g(y))\,dy \qquad \text{(Shell)}$$

$$= \int_0^a 2\pi f(x)\,(a - x)\,f'(x)\,dx \qquad \begin{matrix} y = f(x) \\ dy = f'(x)dx \end{matrix}$$

$$= \int_0^a \pi\,(a-x)\,d[f(x)^2]$$

$$= \Big[\pi(a-x)f(x)^2\Big]_0^a - \int_0^a f(x)^2\,d[\pi(a-x)]$$

$$= \int_0^a \pi f(x)^2\,dx \qquad \text{(Disk)}$$

Infinite Sum (NOT FOR EXAM)

eg $\lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\cdots+\frac{n}{n^2+n^2}\right)=\ ?$

Sol General term is

$$\frac{n}{n^2+k^2}=\frac{1}{n}\,\frac{n^2}{n^2+k^2}=\frac{1}{n}\,\frac{1}{1+\left(\frac{k}{n}\right)^2}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\,\frac{1}{1+\left(\frac{k}{n}\right)^2}$$

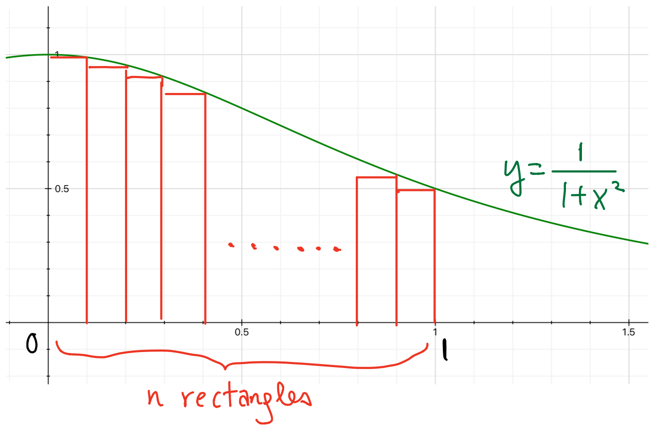
$$=\int_0^1\frac{1}{1+x^2}\,dx$$

$$=\left[\arctan\right]_0^1$$

$$=\arctan 1-\arctan 0$$

$$=\frac{\pi}{4}-0$$

$$=\frac{\pi}{4}$$



Area of k-th rectangle $=\frac{1}{n}\cdot\frac{1}{1+\left(\frac{k}{n}\right)^2}$

In general, for a continuous function $f(x)$

$$\boxed{\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}f\left(\frac{k}{n}\right)=\int_0^1 f(x)\,dx}$$

Ex Find $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k}$ (Ans: $\ln 2$)

Example in Physics : Potential Energy

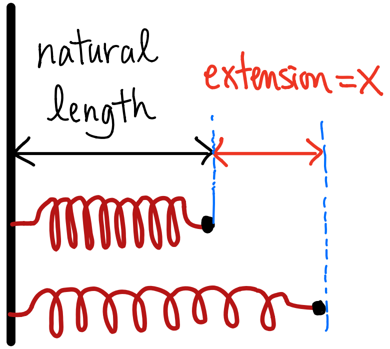
If a constant force F is applied to move an object for a distance s, the work done (energy requird) is

$$W = Fs$$

If F is not a constant but a function of s, then

$$W = \int F(s)\,ds$$

eg When a spring is extended by x it gives a force $F = kx$, where k is a constant



Work done to extend it from $x=0$ to 10:

$$W = \int_0^{10} kx = \left[\tfrac{1}{2}kx^2\right]_0^{10} = 50k$$

The spring gains $50k$ of potential energy.